Continuous model for vocal fold oscillations to study the effect of feedback

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In this work we study the effects of delayed feedback on vocal fold dynamics. To perform this study, we work with a vocal fold model that is made as simple as possible while retaining the spectral content characteristic of human vocal production. Our results indicate that, even with the simplest explanation for vocal fold oscillation, delayed feedback due to reflected sound in the vocal tract can lead to extremely rich dynamics.

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I. INTRODUCTION

The opposed membranes at the base of the human vocal tract are the source of voiced sounds. Airflow induced instability of this structure, known as the vocal folds, modulates the airflow, giving rise to a sequence of pressure pulses which propagate in the vocal tract and are radiated as sound. The modeling of vocal fold oscillations has a long and rich history. The assumptions of modern models of the vocal folds can be traced to the seminal work of Ishizaka and Flanagan [1]. Interested in the problem of achieving a realistic synthesis of voiced sounds, they built a very successful model of two coupled masses, which is used almost 30 years after its publication [2]. Although the authors of this model also mention the importance of understanding the critical parameters of the mechanism in order to address the diagnosis of voice disorders, it has been pointed out that the key parameters in the model have been difficult to relate to anatomical features [3]. Keeping the simplicity of the two-mass model, Story and Titze introduced a three-mass model that allows a better connection between physiological and model parameters [4]. This model builds upon the work of Hirano [5], who stressed the importance of understanding vocal fold tissue properties in order to properly explain the onset of vocal fold oscillations.

In brief summary, the simplest models [6,3] are one-mass models in which the vocal folds are modeled by a one-mass spring driven by airflow with inertial coupling to the vocal tract, i.e., the air column acts like a mass of air that is accelerated and decelerated as a unit. The more complex models [7,8] include several masses. Intermediate between these extremes lies a different approach presented by Titze in [3]. This approach consists in letting the vocal folds support an experimentally observed ''flapping'' motion. With this model, only one second-order equation is needed to derive realistic onset conditions for the oscillations in terms of parameters that could easily be checked through experimental measurements. The model also shows how passive effects in the vocal tract might affect the dynamics of the folds. It should be pointed out that, although nonlinear terms were discussed, large-amplitude oscillations were beyond the interest of the work. In particular, the nonlinear effects due to vocal fold collision were explicitly omitted. Titze's model for flapping motion will be discussed in some detail in the next section.

In all these models, the separation of source and filter is assumed. In this framework, the vocal system is formed by an active nonlinear source of sound (oscillatory glottis) which excites a passive linear filter (vocal tract). The propagation of sound in the filter cannot influence the vocal fold dynamics except passively. In terms of the usual electric circuit analogy, the source-filter separation is based on the hypothesis that the glottis output impedance is much larger than the vocal tract input impedance. The hypothesis holds under normal speech conditions: the mean glottal area is much smaller than the input cross section of the vocal tract, and the fundamental frequency of the glottis is below the first formant (resonance) of the tract [9]. Within the source-filter theory, it is possible to address the existence of subharmonic behavior in vocal production, as in newborn cries [10] and some forms of throat singing [23]. In general, the interpretation of subharmonic effects has been carried out in terms of two-mass models for the vocal folds [2,11].

In this work, the source-filter separation does not apply. We study the effects of delayed feedback on vocal fold dynamics. Feedback arises when the glottal system is coupled to the vocal tract, and pressure reverberations are allowed to go back to the folds and perturb their dynamics after a time delay given by sound speed. The source-filter separation does not hold here, as the glottal oscillations are perturbed by the pressure they created in the vocal tract. The dynamical equations for the glottis and the equations for the pressure in the vocal tract must then be solved simultaneously.

We adopt Titze's flapping model [3]. The flapping model represents a reasonable compromise between a realistic description and complexity, keeping the essentials of vocal fold physics within only one second-order dynamical equation. The motivation of our choice is dynamical in nature. The two- and three-mass models have a phase space dimensionality (four and six, respectively) that allows us to find complex solutions such as period doubling, even disregarding the effect of feedback. Since we are interested in exploring the complexity that the feedback might induce in the dynamics of the folds, it is important, as a first step, to work with a system which, for low coupling, displays a very simple (two-dimensional) dynamics. Differential equations with delay (as will arise in the study of the effect of feedback on the dynamics of the folds) are a complex field within the theory of nonlinear systems [12]. Therefore, we choose to make our first explorations in the field separating the sources of complexity (high dimensionality of the dynamics of the folds, and eventual effects associated with the feedback).

The work is organized as follows. In Sec. II, the flapping model presented by Titze in [3] will be reviewed. In Sec. III, we discuss the dynamical properties of relaxation oscillations, in order to propose reasonable nonlinear dissipation terms that extend the flapping model to study limit cycles beyond the threshold of oscillation. In Sec. IV, the solutions of this extended model in phase space with feedback are discussed. Sec. V contains our conclusions.

II. MODEL FOR VOCAL FOLD OSCILLATIONS

In order to study the physical mechanisms behind small-amplitude oscillations in the vocal folds, Titze proposed a simple model [3]. The idea is that sustained oscillations arise whenever the energy transfer from the airflow to the folds overcomes the dissipative losses. This transfer can be achieved if the driving force exerted by the glottal pressure is larger when the folds are opening than when the folds are approaching each other. Titze observed that this requirement is met when the vocal folds assume an oscillation characterized by a ''flapping'' motion. That is to say, if the vocal cords have a convergent profile while opening and a divergent profile while closing, the pressure on the membranes will gain in energy.

The flapping motion is obtained in a simple way: the vocal folds are assumed to support a longitudinal traveling wave on their surface, in addition to the lateral oscillation of their centers of mass. The assumption of a flapping motion is based on experimental evidence [3,5,22]. Videostroboscopy showed a longitudinal traveling wave propagating on the vocal fold surface, possibly due to a displacement of mucosal tissue traveling upward in the glottis.

As displayed in Fig. 1, a trapezoidal prephonatory glottis is assumed. For small-amplitude oscillations, the model reads

$$Mx''+Kx+Bx'=P_g, \tag{1}$$

where x stands for the departure of the midpoint of the folds from the prephonatory profile and P_g denotes the spatial average of the glottal driving pressure, which has a nonuniform profile. M, K, and B are the mass, stiffness, and damping per unit area of the fold, lumped at the midpoint of the glottis. As stated above, the model further assumes that mucosal tissue surface waves propagate on the vocal fold surface from bottom to top at speed c_w. In this way, a purely kinetic argument lets us write the glottal areas at entry and exit as, respectively,

$$a_1=2L_g(x_{01}+x+\tau x'), \tag{2}$$

$$a_2=2L_g(x_{02}+x-\tau x'), \tag{3}$$

where $2L_g x_{01}$ and $2L_g x_{02}$ are the prephonatory glottal areas at entry and exit, and $\tau\equiv T/(2c_w)$ denotes the time that it takes the surface wave to travel half the way from bottom to top. L_g is the glottal length in the anteroposterior direction (perpendicular to the plane of the paper).

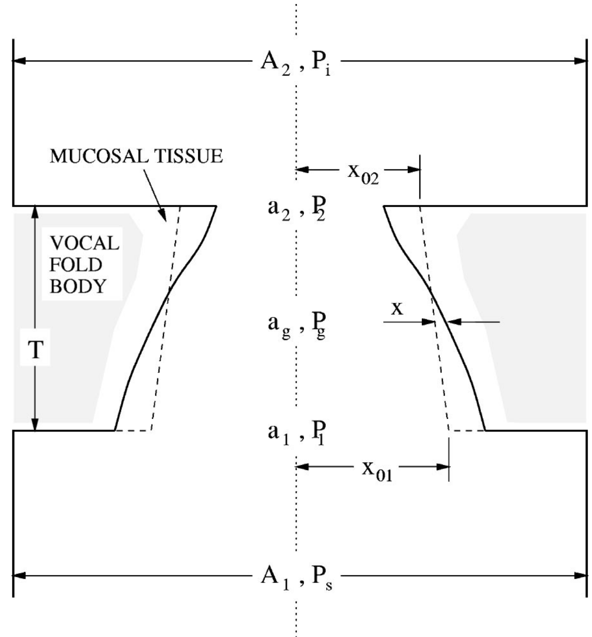

FIG. 1. Frontal section of the flapping model for the vocal folds (taken from [3]). Dashed line: trapezoidal prephonatory glottis. Solid line: surface wave of mucosal tissue traveling upward. T stands for vocal fold thickness.

The last ingredient in the building of this model is the construction of a relationship between the geometric profiles and the pressure. Assuming that the relationship between pressure and profiles of static configurations is a good approximation for oscillatory regimes (see [3]), it can be written that

$$P_g=P_i+(P_s-P_i)(1-a_2/a_1-k_e)/k_t \tag{4}$$

$$=P_i+(P_s-P_i)\left(\frac{\Delta x_0+2\tau x'}{x_{01}+x+\tau x'}\right), \tag{5}$$

where P_s stands for the subglottal (lung) pressure, P_i stands for the input pressure at the vocal tract, $\Delta x_0\equiv x_{01}-x_{02}$, and $k_e\approx 0.1$, $k_t\approx 1.1$ are phenomenological coefficients.

Computing the stationary solutions of the resulting dynamical equation, and linearizing around the physically meaningful one, it is possible to write the analytical conditions in which sustained oscillations arise. This occurs when the negative dissipation induced by the flow [measured by the coefficient of x' after expanding Eq. (5)] overcomes the viscous dissipation of the system (controlled by the parameter B). For realistic parameter values, the oscillation threshold for the lung pressure reads

$$P_L^{th}\approx\left(\frac{x_{01}^2}{2x_{01}-\Delta x_0}\right)\frac{B}{\tau}. \tag{6}$$

Simple as it is, this model achieves a reasonable prediction of the onset conditions for oscillations in terms of physically measurable parameters [3,13], and a simple mathematical framework to describe the basic mechanisms involved.

The model, however, was not conceived to work beyond small oscillations: the saturation mechanisms responsible for stopping the folds at the returning points are absent. Mathematically, almost any numerical integration will give rise to a collision of the trajectory with the singular line in phase space defined by $x_{01}+x+\tau x'=0$. The exceptions involve initial conditions in the one-dimensional manifold tending to the point in the singular line in which $\Delta x_0+2\tau x'=0$.

Numerical simulations of more complex models [14] reproduce adequately the characteristic skewness of glottal wave forms. This skewness can be described as a second time derivative of the wave form larger (smaller) than zero for most of the opening (closing) phase. Therefore the question arises: is it possible to dress this simple model, which accounts reasonably for the onset conditions of oscillations, with nonlinear saturation terms that extend the model to include the returning points of the oscillation within a continuous dynamical system? Is it possible to reproduce with this dressed model the qualitative shape of the glottal oscillations? To accomplish our task, we need to introduce continuous nonlinear dissipations that allow us to explore Titze's flapping model beyond the onset. These terms will, of course, be caricatures of the nonlinear dissipations that are needed to model fold collisions. In order to answer these questions we will present a brief review of relaxation oscillations, which can be omitted by the expert reader.

III. RELAXATION OSCILLATIONS

What are the minimal ingredients needed to achieve successive jumps between two states (e.g., ''open'' and ''closed'') with a continuous model? One of the most important achievements of nonlinear dynamics has been the presentation of paradigmatic equations with solutions that are able to mimic a dynamical property of interest. A typical example is the mechanism known as ''relaxation oscillation,'' conceived to model precisely successive jumps between two states.

In order to model an oscillation in terms of a dynamical system, at least two variables are needed. What characterizes a relaxation oscillation is the existence of two different time scales for the dynamics of each variable. The simplest example was provided by van der Pol in 1926 [15]. It involves two dynamical variables u and v, related through the following equations:

$$u'=v-u^3+u, \tag{7}$$

$$v'=-\epsilon u, \tag{8}$$

where ϵ is a small real parameter. The dynamics is simple to understand in the case $\epsilon=0$: the variable v becomes a parameter, and the bifurcation diagram for the first equation is displayed in Fig. 2(a). The fixed points in the solid line branches are stable, while the fixed points in the dashed line branch are unstable. Therefore, in this limit, the variable u (known as the *fast* variable) relaxes to one of the stable states, determined by the (constant) value of v. For $0<\epsilon \ll 1$, however, the system slowly moves close to the curve given by $v=u^3-u$ (which is the solution in the case $\epsilon=0$).

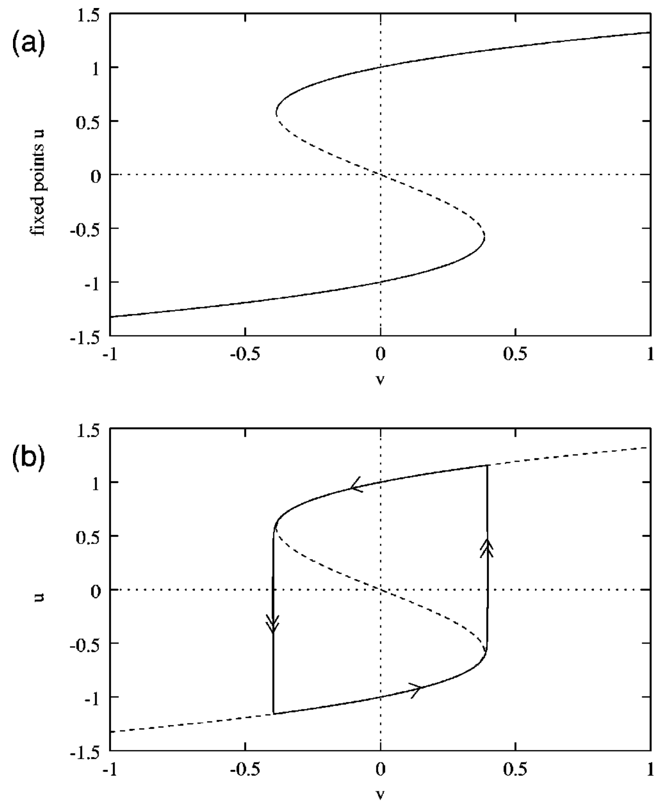

FIG. 2. Relaxation oscillator. (a) Bifurcation diagram for Eq. (7) in the case $\epsilon=0$: fixed points in variable u as a function of v (treated as a parameter). Stable fixed points are represented by solid lines, unstable fixed points by dashed lines. (b) Phase space for the relaxation oscillator, Eqs. (7) and (8).

As shown in Fig. 2(b), when the trajectory is close to the lower branch v slowly increases as a function of time, until the tangent of the branch is vertical. At this point, the system rapidly jumps close to the upper branch, and v begins to slowly decrease with time. A second jump occurs when the tangent to the upper branch is vertical, this time to the lower branch, and the system continues alternating between ''upper'' and ''lower'' states.

If a model is built from Newton's laws, it will naturally be stated in terms of a second-order equation:

$$x''=f(x,x',t). \tag{9}$$

It is natural then to ask what kind of nonlinear terms (and parameter values) are needed to obtain relaxation oscillations in a second-order dynamical system. Let us identify the variable u in the relaxation oscillator with the variable x in our dynamical system Eq. (9). By virtue of Eq. (7), its time derivative is $x'=v-u^3+u$. Therefore, its second time derivative is $x''=v'-3u^2u'+u'$. In this way, our second-order dynamical system reads

$$x''=-\epsilon x-3x^2x'+x'. \tag{10}$$

The interpretation of this equation is clear in physical terms: it describes the dynamics of a particle subjected to a small restitution term $-\epsilon x$, a *linear* positive dissipation

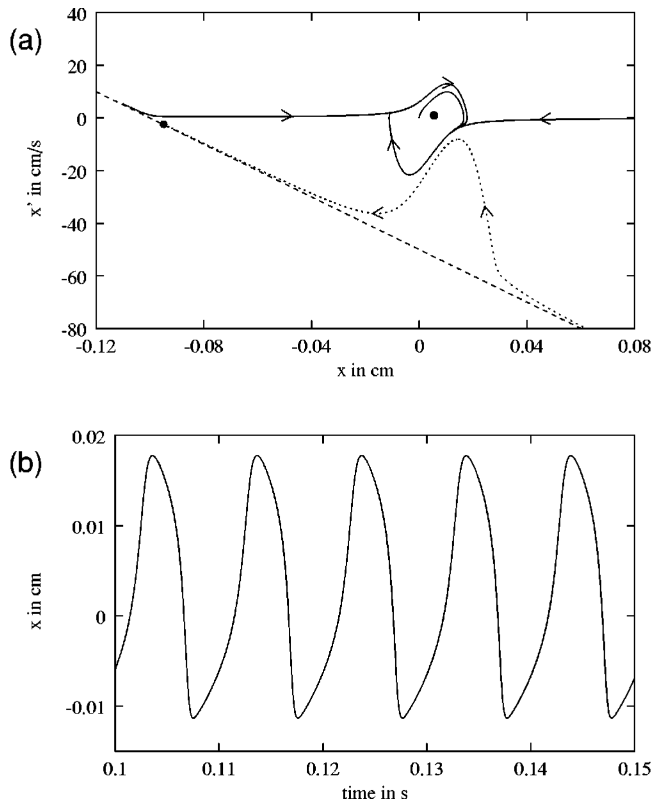

FIG. 3. Extended model, without coupling to the vocal tract. (a) Phase space. Dashed straight line: singular line. Dotted line: trajectory tending to the special point on the singular line. Solid line: limit cycle and main trajectories tending to it. (b) Corresponding time series of x (limit cycle). Fundamental frequency of oscillation is $F_0=100$ Hz. Parameter values are lung pressure $P_L=15\,000$ dyn/cm^2, stiffness $K=250$ kdyn/cm^3, damping coefficients $B=100$ (dyn segment)/cm^3 and $C=10^5$ cm^{-2}, vocal fold mass $M=0.45$ g/cm^2, prephonatory positions $x_{01}=0.1$ cm and $x_{02}=0.09$ cm, vocal fold length $L_g=1.4$ cm (anteroposterior direction), vocal fold thickness $T=0.4$ cm (along the flow), surface wave velocity $c_w=100$ cm/s, and $\tau\equiv T/(2c_w)=2$ ms.

$f_{ext}=x'$, and a negative dissipation $f_d=-3x^2x'$. The linear positive dissipation term represents an external force in phase with the velocity, accounting for the transfer of energy to the system, while the negative dissipation term becomes important as the dynamical variable x travels far from the position of equilibrium $\overline{x}\equiv 0$. This nonlinear term is responsible for the saturation that avoids a divergence of the flow, since the negative linear dissipation causes a loss of stability of the fixed point. With this term, it is possible to generate a wave form with a second time derivative larger (smaller) than zero for most of the opening (closing) phase [see Fig. 3(b) below].

IV. THE EXTENDED MODEL

In this section we will analyze the structure of the solutions of Titze's flapping motion model [3], extended with a nonlinear dissipative force $f_d=B[1+C(x-\overline{x})^2]x'$. As stated in the previous section, this term accounts for a high dissipation whenever the absolute value of the departure from the stationary position $\overline{x}$ is large. In the first place, we begin by inspecting the structure of the solutions of this model without coupling to the tract [i.e., we will consider that the vocal tract input pressure in Eq. (5) is the atmospheric pressure, $P_i=0$]. Equation (1) therefore will read

$$Mx''+Kx+B[1+C(x-\overline{x})^2]x'=\frac{\Delta x_0+2\tau x'}{x_{01}+x+\tau x'}P_L. \tag{11}$$

Clearly, the nonlinear dissipation is negligible at the onset of the oscillations. But once the fixed point loses its stability a limit cycle is observed. In Fig. 3(a), several trajectories are displayed. The dashed straight line is the singular line in which $x_{01}+x+\tau x'=0$. There is a point on this singular line in which the numerator of the forcing term is also zero: $\Delta x_0+2\tau x'=0$; the dotted line represents a one-dimensional trajectory that tends to this singular point. This trajectory defines the boundary of the basin of attraction of the limit cycle. Dynamically, the singular point behaves as a saddle point and, by reducing the value of B, we can increase the size of the limit cycle, reducing its distance in phase space from the singular point as much as we want. The corresponding time trace of x is displayed in Fig. 3(b).

Until this point, the glottal system has been considered open to the atmosphere, i.e., $P_i=0$. A more accurate picture is the following: Fluctuations in the glottal flow U induce fluctuations in the pressure at the input of the vocal tract, and forward waves propagate this perturbation. With various time delays, the reflections of the propagating sound wave return to the base of the vocal tract, and add to the original perturbations of pressure. To account for this feedback effect, the solution of the problem requires a treatment in which both the source system and the passive filter are simultaneously solved.

At frequencies below the first formant of the vocal tract, the source of pressure perturbations at the input of the vocal tract $s(t)$ is proportional to the time variation of the glottal flow $U(x)$, through a coefficient called inertance [18]. This is Newton's law for the vocal tract air column:

$$s(t)=I\frac{dU}{dt}, \tag{12}$$

where the inertance is $I=\rho L/A$, L and A being vocal tract length and cross section, respectively (ρ is air density). When a pressure wave propagates through a vocal tract whose cross section varies along its length, then the pressure wave passes through a medium whose inertance locally depends on the cross section of the vocal tract. Variations in inertance give rise to reflections at the boundary between sections of different inertance. To reproduce vowels, which depend on articulation of the shape of the vocal tract, it is customary to approximate the tract by a set of tubes (at least two). In this situation, the pressure wave generated at the input of the vocal tract is partially reflected and partially transmitted at each interface between consecutive tubes. At the interface between the last tube and the atmosphere, the wave is partially reflected and partially emitted toward the atmosphere.

For the sake of simplicity, in this work we set the vocal tract to be uniform. That is, the vocal tract is modeled by a single tube of uniform diameter. Although unrealistic, it is the simplest choice to account for the effects of feedback with vocal tract coupling. Calling $a(t)$ and $b(t)$ the forward and backward pressure waves in the vocal tract (respectively), the equations accounting for the boundary conditions are

$$P_i(t) \equiv a(t) = s(t) + b(t-\tau_1), \tag{13}$$

$$b(t) = \gamma a(t-\tau_1), \tag{14}$$

where γ accounts for the reflection coefficient of the interface between the end of the vocal tract and the atmosphere (if no losses, $\gamma = -1$), and $\tau_1 \equiv L/c_s$ is the time it takes for a sound wave to travel the vocal tract length L [16,17].

In this way, we can study the effect of feedback on the oscillations of the glottis. Notice that this treatment departs from the source-filter theory of voiced sound production. Since feedback is neglected without great detriment in the models of speech described in the Introduction, it is important to confirm that for realistic parameter values of normal speech the effect of feedback is negligible. We expect, for small inertance (i.e., low coupling, glottis connected to a wide tube), that the input pressure fluctuations in the tract will not qualitatively affect the dynamics of the glottis. The result will be, in our simple model, an oscillation of the average glottal size which can be embedded in two dimensions. A different scenario will occur for high values of the coupling between the glottis and the tube. The input pressure will not only be high: it will be the result of adding the source fluctuations and the waves that are reflected at the interfaces of the vocal tract. Therefore, complex solutions of delay differential equations can be expected to occur [19].

In order to examine the effect of feedback on the glottal oscillations, we inspect the projection of the glottal dynamics onto the (x,x') coordinates. For low values of the coupling between the source and the vocal tract, a simple oscillation will take place. We then take a Poincaré section [12], and check if for higher values of the coupling more than one intersection occurs. We call the number of intersections of the flow with the Poincaré plane the *period* of the glottal solution.

Fig. 4 displays the results of integrating our model for different values of the parameters L,I describing the uniform vocal tract. The y axis indicates the degree of coupling (inertance) and the x axis is inversely proportional to the frequency of the first formant of the tube, while the shading level stands for the period of the orbit. This choice of parameter representation is inspired by the problem of forced nonlinear oscillators, for which different regions of parameters space correspond to different lockings between the forcing and the autonomous oscillator [20]. In this representation, we do find structures that resemble this dynamical scenario. For low values of the coupling, the system is in what we call a period-1 state: the projection of the glottal dynamics onto the (x,x') plane is a non-self-intersecting curve. That is, at normal speech values for the inertance ($I<0.06$ g/cm^4, i.e., vocal tract cross section $A>0.3$ cm^2) a period-one oscillation of normal speech is recovered for a realistic $L=17.5$ cm. Note that $A \approx 0.3$ cm^2 is a typical value for normal speech for the epilarynx tube, the first section of the vocal tract [21].

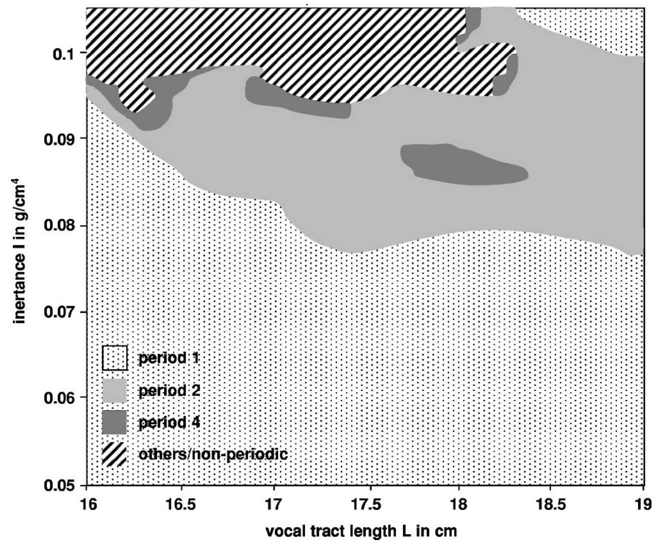

FIG. 4. Extended model, with coupling to the vocal tract and feedback. Regions of periodicity in (L,I) parameter space. The x axis (vocal tract length L) is inversely proportional to the frequency of the first formant of the tube. Lengthening (shortening) of the tube brings the formants toward lower (higher) frequencies. The y axis (vocal tract inertance I) stands for the degree of coupling. A stronger coupling to the vocal tract is achieved through a narrowing of the tube. Parameter values are the same as in Fig. 3, except for $\tau=0.2$ ms; in addition, $\gamma=-0.9$, air density $\rho=0.001\,14$ g/cm^3, and sound velocity $c_s=35\,000$ cm/s. The regions in parameter space correspond to values in which solutions of period 1, 2, 3, and 4 can be found. The large region labeled *others/nonperiodic* corresponds to values of the parameters where periodic solutions of period larger than 4 and nonperiodic solutions can be found.

As coupling is increased, we find frequency ranges for which subharmonic glottal oscillations occur. The largest region of subharmonic solutions is one for which a period-2 solution arises. This solution is born in a period doubling bifurcation: no matter where we choose to take a Poincaré section the flow intersects it at two points. In Fig. 5(a) we display an oscillation in the (x,x') space for the period-2 region, at the values $L=17.5$ cm and $I=0.081$ g/cm^4 (i.e., vocal tract cross section $A=0.25$ cm^2, about 80% of a normal area). The time series of x is displayed in Fig. 5(b). In Fig. 5(c), we show the pressure fluctuations at the exit of the vocal tract as a function of time. Notice that beyond the supraharmonics (small oscillations mounted on the one corresponding to the fundamental frequency), the halved-frequency subharmonic can be seen by simple inspection, in either Fig. 5(b) or Fig. 5(c). For these parameter values, the pressure fluctuations due to feedback account for approximately 30% of the glottal pressure.

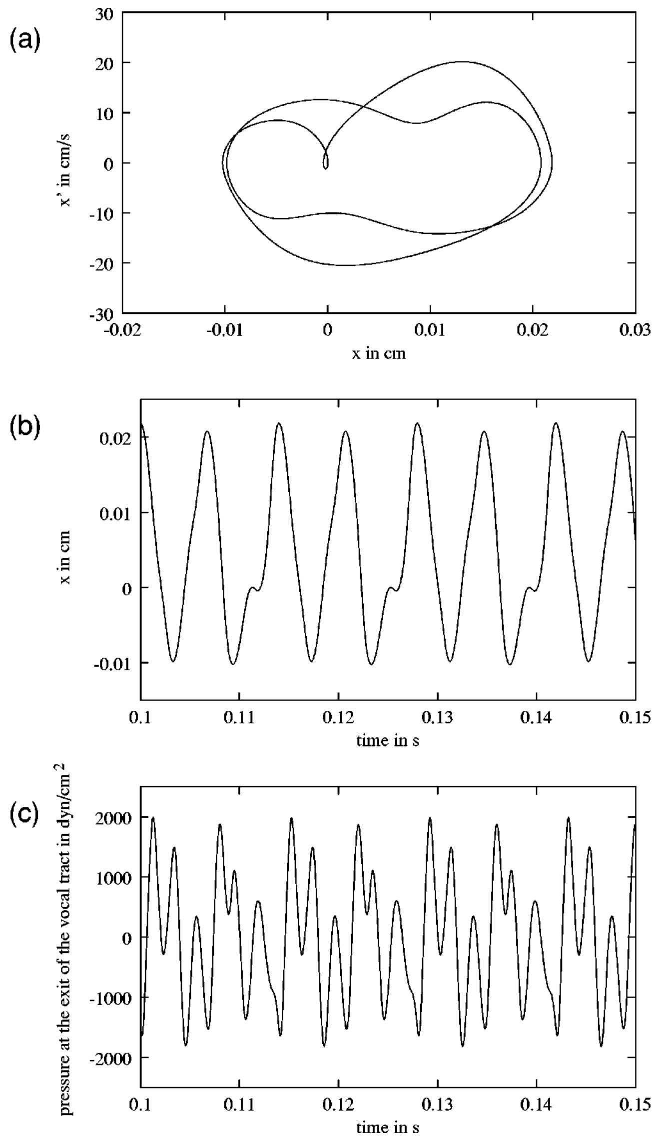

FIG. 5. Period-2 solution, for parameter values $L=17.5$ cm and $I=0.081$ g/cm^4 from Fig. 4. Fundamental frequency of oscillation (halved at the bifurcation) is $F_0/2=72$ Hz. (a) Limit cycle in (x,x') space. (b) Corresponding time series of x. A similar ripple structure is reported experimentally in [22]. (c) Corresponding time series of the pressure at the exit of the vocal tract.

For a vocal tract length slightly different (and higher values of the coupling), period-4 solutions bifurcate from the period-2 ones. We report also regions of period-3 and period-6 solutions (not shown here, out of scale).

The existence of period doubling bifurcations has already been reported in the literature of the human voice. In [22], subharmonic responses were reported as a male speaker was instructed to perform a given vocal maneuver. A ripple structure similar to the one shown in Fig. 5(b) was observed in the subharmonic cycle of the folds. Yet, in this case, the videostroboscopic view of the cycle showed a left-right difference between the folds: an asymmetrical motion, possibly causing the bifurcation. Another example of period doubling bifurcation in the human voice was reported in the Kargyraa style of harmonic chant (for a review, see [23]). In this case, the second frequency was associated with a second source of vibration (the false vocal folds). With the subharmonic tongues displayed in Fig. 4, delayed feedback would then be a third possible mechanism for subharmonic solutions of the vocal fold dynamics.

It is possible to go beyond the numerical simulation to understand the nature of the mechanism behind the creation of these solutions once the coupling between the glottis and the vocal tract is introduced. Notice that the boundary conditions (14) are effectively acting as delay terms in the differential equation. It is possible then to study the linear loss of stability of the equilibrium position of the glottis, and Hopf-Hopf interactions will be found. Solutions like the ones reported in this work were reported in a similar dynamical scenario [19].

V. CONCLUSIONS

In this work, we have analyzed a simple model for voice production. This model is based on the flapping model proposed in [3]. Nonlinear dissipative terms were introduced to mimic the high dissipation at large values of departure from equilibrium, with a choice of coefficients that reproduce the characteristic shape of the dynamics of the folds. This model needs no coupling to the tract to sustain oscillations. In this situation, its dynamics is very simple: at most, nonlinear relaxation oscillations are present. On the other hand, this model allows us to study in a very natural way the effect of delayed feedback when the glottis and the vocal tract are coupled. Our explorations indicate that, in solving simultaneously the dynamical equations of the folds and the pressure reverberations in the tract, interesting dynamics can be found. The organization of solutions strongly resembles what is observed in periodically forced nonlinear oscillators: tongues of subharmonic solutions can be found in a parameter space which accounts for the degree of coupling and a characteristic time of the ‘‘forcing.’’

The production of voiced sounds is a very complex problem which displays many interesting phenomena of fluid dynamics. Therefore, any low-dimensional model implies strong approximations. The spirit of this work is to study the effect of feedback in a system which is, before the coupling to the vocal tract, as simple as possible. Our results indicate that even in this case the dynamics can be extremely rich. Many situations have been reported in which subharmonic behavior exists in vocal production, from newborn cries to some styles of throat singing. It is not clear that in these cases the effect of feedback alone can explain the behavior. It should always be kept in mind that models of several masses as well as models based on the nonlinear interaction between modes of the folds can generate subharmonic solutions, even in the absence of feedback. In this work, we found that the effect of feedback alone is enough, for reasonable parameter values, to introduce a rich variety of subharmonic solutions.

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